Supplemental Material I: Technical Details

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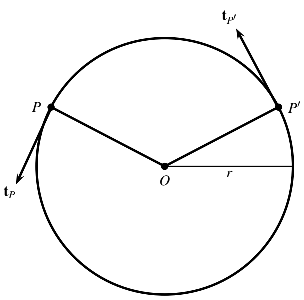
1 Reconstruction by floating tangents interpolation

This section gives the mathematical details of the reconstruction by floating tangents interpolation (section 5.2 in the article). We start by the proof of lemma 5.1 that gives a relation between two adjacent gate points. Then we give the linear system we use to solve the general case.

1.1 Proof of lemma 5.1

Lemma 1.1 (Preliminary lemma)**.** *Let $(\mathscr{C})$ be a circle centered in O with radius r. If $\mathbf{t}_P$ is the normalized and oriented tangent to $(\mathscr{C})$ at P, the cross product $OP \times \mathbf{t}_P$ is constant when P follows the same direction on the circle.*

Proof. When P is on the circle, $OP = \pm r\overline{\mathbf{t}_P}$ whether the direction is counterclockwise or not. Thus, $OP \times \mathbf{t}_P$ is equal to $\pm r\mathbf{e_z}$ and is constant when P follows the same direction on the circle.

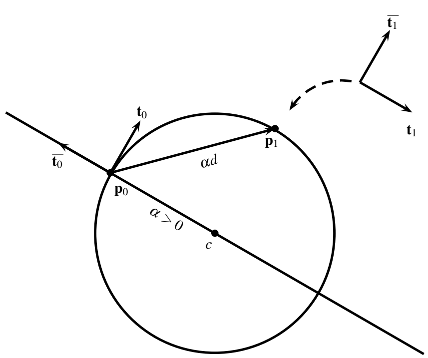


$det(OP, \mathbf{t}_P) = det(OP', \mathbf{t}_{P'}) = -r$

□

Let $\mathbf{p}_0$ be a point with a given tangent $\mathbf{t}_0$. Let $(\mathscr{C})$ be a circle going through $\mathbf{p}_0$ with the tangent $\mathbf{t}_0$. Then the center c of $(\mathscr{C})$ is on the line (D) of direction $\overline{\mathbf{t}_0}$ and going through $\mathbf{p}_0$. Let α be a scalar so that $c = \mathbf{p}_0 - \alpha\overline{\mathbf{t}_0}$. The radius of $(\mathscr{C})$ is then $|\alpha|$. If the point $\mathbf{p}_1$ is on $(\mathscr{C})$ with the tangent $\mathbf{t}_1$, then $\mathbf{p}_1 = c \pm \alpha\overline{\mathbf{t}_1}$. The preliminary lemma actually shows that $\mathbf{p}_1 - c = \alpha\overline{\mathbf{t}_1}$ since $cp_1 \times \mathbf{t}_1 = \pm\alpha = cp_0 \times \mathbf{t}_0$, so $\mathbf{p}_1 = c + \alpha\overline{\mathbf{t}_1}$ and finally, $\mathbf{p}_1 = \mathbf{p}_0 + \alpha(\overline{\mathbf{t}_1} - \overline{\mathbf{t}_0})$.

Conversely, let $\mathbf{t}_0$ and $\mathbf{t}_1$ be two different vectors of length 1, and $\mathbf{d}$ built according to the lemma. Let $\mathbf{p}_0$ be a point and $\mathbf{p}_1 = \mathbf{p}_0 + \alpha\mathbf{d}$ with $\alpha \in \mathbb{R} - \{0\}$. Then, let $c = \mathbf{p}_0 - \alpha\overline{\mathbf{t}_0}$. Thus $\mathbf{p}_0 = c + \alpha\overline{\mathbf{t}_0}$ and $\mathbf{p}_1 = c + \alpha\overline{\mathbf{t}_0} + \alpha\mathbf{d}$ so $\mathbf{p}1 = c + \alpha\overline{\mathbf{t}_0} + \alpha\overline{\mathbf{t}_1} - \alpha\overline{\mathbf{t}_0}$ and finally $\mathbf{p}_1 = c + \alpha\overline{\mathbf{t}_1}$. Then the circle centered in c with radius $|\alpha|$ goes through $\mathbf{p}_0$ and $\mathbf{p}_1$ with the tangents $\mathbf{t}_0$ and $\mathbf{t}_1$. The preliminary lemma insures that this circle is well-oriented.



1.2 Linear system resolution

In this section, we give the linear system used to solve the general case of the reconstruction by floating tangents interpolation. Recall that

$$f(\mathbf{p}'_0, \alpha_0, \ldots, \alpha_{N-1}) = \sum_{i=0}^{N} ||\mathbf{p}'_0 + \sum_{k=0}^{i-1} \alpha_k \mathbf{d}_k - \mathbf{p}_i||^2.$$

We are looking for $\mathbf{p}'_0, \alpha_0, \ldots, \alpha_{N-1}$ so that

$$\frac{\partial f}{p'_{0x}} = 0 \quad \frac{\partial f}{p'_{0y}} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha_j} = 0 \quad \forall j \in \{0, \ldots, N-1\}.$$

The derivatives are:

$$\frac{\partial f}{\partial \mathbf{p}'_{0x}} = 2(N+1)\mathbf{p}'_{0x} + 2\sum_{k=0}^{N-1}(N-k)\alpha_k \mathbf{d}_{kx} - 2\sum_{k=0}^{N}\mathbf{p}_{ix},$$

$$\frac{\partial f}{\partial \mathbf{p}'_{0y}} = 2(N+1)\mathbf{p}'_{0y} + 2\sum_{k=0}^{N-1}(N-k)\alpha_k \mathbf{d}_{ky} - 2\sum_{k=0}^{N}\mathbf{p}_{iy},$$

and

$$\begin{aligned} \frac{\partial f}{\partial \alpha_l} =& 2(N-l)\mathbf{d}_{lx}\mathbf{p}_{0x} + 2\mathbf{d}_{lx}\sum_{k=0}^{l}(N-l)\alpha_k\mathbf{d}_{kx} \\ &+ 2\mathbf{d}_{lx}\sum_{k=l+1}^{N}(N-k)\alpha_k\mathbf{d}_{kx} - 2\mathbf{d}_{lx}\sum_{i=0}^{N-1}\mathbf{p}_{ix} \\ &+ 2(N-l)\mathbf{d}_{ly}\mathbf{p}_{0y} + 2\mathbf{d}_{ly}\sum_{k=0}^{l}(N-l)\alpha_k\mathbf{d}_{ky} \\ &+ 2\mathbf{d}_{ly}\sum_{k=l+1}^{N-1}(N-k)\alpha_k\mathbf{d}_{ky} - 2\mathbf{d}_{ly}\sum_{i=0}^{N}\mathbf{p}_{iy}. \end{aligned}$$

Thus, we obtain the linear system

$$\mathbb{A}\begin{pmatrix} \mathbf{p}'_{0x} \\ \mathbf{p}'_{0y} \\ \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} 2\sum_{k=0}^{N}\mathbf{p}_{ix} \\ 2\sum_{k=0}^{N}\mathbf{p}_{iy} \\ 2\mathbf{d}_{0x}\sum_{k=0}^{N}\mathbf{p}_{ix} + 2\mathbf{d}_{0y}\sum_{k=0}^{N}\mathbf{p}_{iy} \\ \vdots \\ 2\mathbf{d}_{(N-1)x}\sum_{k=0}^{N}\mathbf{p}_{ix} + 2\mathbf{d}_{(N-1)y}\sum_{k=0}^{N}\mathbf{p}_{iy} \end{pmatrix}$$

in which the first row of $\mathbb{A}$ is

$$\begin{pmatrix} 2(N+1) & 0 & 2N\mathbf{d}_{0x} & \cdots & 2(N-k)\mathbf{d}_{kx} & \cdots & 2\mathbf{d}_{(N-1)x} \end{pmatrix},$$

the second one is

$$\begin{pmatrix} 0 & 2(N+1) & 2N\mathbf{d}_{0y} & \cdots & 2(N-k)\mathbf{d}_{ky} & \cdots & 2\mathbf{d}_{(N-1)y} \end{pmatrix},$$

and finally the row l is

$$\begin{aligned} \mathbb{A}_{l,0} &= 2(N-l)\mathbf{d}_{lx}, \\ \mathbb{A}_{l,1} &= 2(N-l)\mathbf{d}_{ly}, \\ \mathbb{A}_{l,k} &= 2\mathbf{d}_{lx}(N-l)\mathbf{d}_{kx} + 2\mathbf{d}_{ly}(N-l)\mathbf{d}_{ky} \quad \text{if} \quad k \leq l, \\ \mathbb{A}_{l,k} &= 2\mathbf{d}_{lx}(N-k)\mathbf{d}_{kx} + 2\mathbf{d}_{ly}(N-k)\mathbf{d}_{ky} \quad \text{if} \quad k > l. \end{aligned}$$

2 Discrete potential energy and derivatives

In this section, we study the discrete potential energy of a super-circle and calculate its Jacobian and its Hessian (see section 6 in the article).

2.1 Discrete potential energy

The discrete potential energy of a super-circle under gravity is given by

$$E_p = E_g + E_{el}$$

with

$$E_{el} = \frac{EI}{2}\int_0^L \left(\kappa(s)-\kappa^0(s)\right)^2 \mathrm{d}s,$$
$$E_g = \rho S g \int_0^L (L-s) sin(\theta(s))\mathrm{d}s.$$

In our case θ is a continuous piecewise linear function of s. By denoting $\theta_i = \theta(s_i)$ we have

$$E_g = \rho g S \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} (L-s) sin(\theta_i + \kappa_i s)\mathrm{d}s.$$

With the substitution $\theta_i' = \theta_i + \kappa_i s$, we get

$$E_g = \rho g S \sum_{i=0}^{n-1} \left[\frac{1}{\kappa_i}\left((L-s_i)cos(\theta_i) - (L-l_i-s_i)cos(\theta_i+\kappa_i l_i)\right) - \frac{1}{\kappa_i^2}\left(sin(\theta_i+\kappa_i l_i) - sin(\theta_i)\right)\right].$$

Finally, the potential energy of the rod is

$$E_p = \sum_{i=0}^{n-1} \frac{EI}{2}\left(\kappa_i - \kappa_i^0\right)^2 l_i + \rho g S \sum_{i=0}^{n-1} \left[\frac{1}{\kappa_i}\left((L-s_i)cos(\theta_i) - (L-l_i-s_i)cos(\theta_i+\kappa_i l_i)\right) - \frac{1}{\kappa_i^2}\left(sin(\theta_i+\kappa_i l_i) - sin(\theta_i)\right)\right].$$

2.2 Jacobian

We have

$$\frac{\partial E_p}{\partial \kappa_j} = \sum_{i=1}^{N} \frac{\partial E_{eli}}{\partial \kappa_j} + \frac{\partial E_{gi}}{\partial \kappa_j}.$$

Given the previous formulations of the elastic energy we see that

if $i \neq j$

$$\frac{\partial E_{eli}}{\partial \kappa_j} = 0,$$

if $i = j$

$$\frac{\partial E_{eli}}{\partial \kappa_i} = EI(\kappa_i - \kappa_i^0) l_i.$$

For the potential gravitational energy, denoting $\theta_i = \theta_0 + \sum_{j=0}^{i-1} \kappa_j l_j$, we have

if $i < j$

$$\frac{\partial E_{gi}}{\partial \kappa_j} = 0,$$

if $i = j$

$$\frac{\partial E_{gi}}{\partial \kappa_j} = \rho g S \left[\frac{1}{\kappa_i}\left((L-l_i-s_i) l_i sin(\theta_i+\kappa_i l_i)\right) - \frac{1}{\kappa_i^2}\left((L-s_i)cos(\theta_i) - (L-2l_i-s_i)cos(\theta_i+\kappa_i l_i)\right) + \frac{2}{\kappa_i^3}\left(sin(\theta_i+\kappa_i l_i) - sin(\theta_i)\right)\right],$$

if $i > j$

$$\frac{\partial E_{gi}}{\partial \kappa_j} = l_j \rho g S \left[\frac{1}{\kappa_i}\left((L-l_i-s_i)sin(\theta_i+\kappa_i l_i) - (L-s_i)sin(\theta_i)\right) - \frac{1}{\kappa_i^2}\left(cos(\theta_i+\kappa_i l_i) - cos(\theta_i)\right)\right].$$

Finally, we have

$$\frac{\partial E_p}{\partial \kappa_j} = \frac{\partial E_{elj}}{\partial \kappa_j} + \sum_{i=j}^{N} \frac{\partial E_{gi}}{\partial \kappa_j}.$$

So the values of κ_j^0 we are looking for are

$$\kappa_j^0 = \kappa_j + \frac{1}{EIl_j}\sum_{i=j}^{N} \frac{\partial E_{gi}}{\partial \kappa_j}.$$

2.3 Hessian

We have

$$\frac{\partial^2 E_p}{\partial \kappa_j \partial \kappa_m} = \frac{\partial^2 E_{elj}}{\partial \kappa_j \partial \kappa_m} + \sum_{i=j}^{n-1} \frac{\partial^2 E_{gi}}{\partial \kappa_j \partial \kappa_m}.$$

If $m \neq j$, we have

$$\frac{\partial^2 E_{elj}}{\partial \kappa_j \partial \kappa_m} = 0,$$

else, we have

$$\frac{\partial^2 E_{elj}}{\partial {\kappa_j}^2} = EIl_j.$$

Regarding the potential gravitational energy, if $i = j = m$

$$\frac{\partial^2 E_{gi}}{\partial {\kappa_i}^2} = \rho g S \Big[\frac{1}{\kappa_i}\left((L-l_i-s_i) l_i^2 cos(\theta_i+\kappa_i l_i)\right) - \frac{1}{\kappa_i^2}\left(l_i(L-l_i-s_i)sin(\theta_i+\kappa_i l_i) - l_i^2 sin(\theta_i+\kappa_i l_i) + l_i(L-s_i-l_i)sin(\theta_i+\kappa_i l_i)\right) + \frac{2}{\kappa_i^3}\left(l_i cos(\theta_i+\kappa_i l_i) - (L-l_i-s_i)cos(\theta_i+\kappa_i l_i) + (L-s_i)cos(\theta_i) + l_i cos(\theta_i+\kappa_i l_i)\right) + \frac{6}{\kappa_i^4}\left(sin(\theta_i) - sin(\theta_i+\kappa_i l_i)\right)\Big],$$

if $i = j > m$

$$\frac{\partial^2 E_{gi}}{\partial \kappa_i \partial \kappa_m} = \rho g S \left[\frac{l_m}{\kappa_i}\left((L-l_i-s_i) l_i cos(\theta_i+\kappa_i l_i)\right) + \frac{l_m}{\kappa_i^2}\left((L-s_i)sin(\theta_i) - (L-2l_i-s_i)sin(\theta_i+\kappa_i l_i)\right) + \frac{2l_m}{\kappa_i^3}\left(cos(\theta_i+\kappa_i l_i) - cos(\theta_i)\right)\right],$$

if $i = m > j$

$$\frac{\partial^2 E_{gi}}{\partial \kappa_i \partial \kappa_m} = \rho g S \left[\frac{l_j}{\kappa_i}\left((L-l_i-s_i) l_i cos(\theta_i+\kappa_i l_i)\right) + \frac{l_j}{\kappa_i^2}\left((L-s_i)sin(\theta_i) - (L-2l_i-s_i)sin(\theta_i+\kappa_i l_i)\right) + \frac{2l_j}{\kappa_i^3}\left(cos(\theta_i+\kappa_i l_i) - cos(\theta_i)\right)\right],$$

and if $i > j$ and $i > m$

$$\frac{\partial^2 E_{gi}}{\partial \kappa_j \partial \kappa_m} = l_j l_m \rho g S \left[\frac{1}{\kappa_i}\left((L-l_i-s_i)cos(\theta_i+\kappa_i l_i) - (L-s_i)cos(\theta_i)\right) + \frac{1}{\kappa_i^2}\left(sin(\theta_i+\kappa_i l_i) - sin(\theta_i)\right)\right].$$

So, finally,

$$\frac{\partial^2 E_p}{\partial {\kappa_j}^2} = EIl_j + \sum_{i=j}^{n-1} \frac{\partial^2 E_{gi}}{\partial {\kappa_j}^2},$$

$$\frac{\partial^2 E_p}{\partial \kappa_j \partial \kappa_m} = \sum_{i=j}^{n-1} \frac{\partial^2 E_{gi}}{\partial \kappa_j \partial \kappa_m} \qquad \text{if} \quad j > m,$$

$$\frac{\partial^2 E_p}{\partial \kappa_j \partial \kappa_m} = \sum_{i=j}^{n-1} \frac{\partial^2 E_{gi}}{\partial \kappa_j \partial \kappa_m} = \sum_{i=m}^{n-1} \frac{\partial^2 E_{gi}}{\partial \kappa_j \partial \kappa_m} \qquad \text{if} \quad m > j.$$