A Nonsmooth Newton Solver for Capturing Exact Coulomb Friction in Fiber Assemblies

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Motivation

- Fibers assemblies are common in the real world
- But not much studied in the past
- Contact and dry friction play a major role w.r.t. shape and motion
  (volume, stable stacking, nonsmooth patterns, nonsmooth dynamics)
Fibers assemblies: Previous work

Main motivation

Hair simulation in Computer Graphics
Main motivation

Hair simulation in Computer Graphics

Three families of models
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Hair simulation in Computer Graphics

Three families of models

1. Continuum-based [Hadap and Magenat-Thalmann 2001]
   → Hair medium governed by fluid-like equations
Main motivation

Hair simulation in Computer Graphics

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   → Hair medium governed by fluid-like equations
   🧡 Macroscopic, intrinsic interaction model
Fibers assemblies: Previous work

Main motivation

Hair simulation in Computer Graphics

Three families of models

1. Continuum-based [Hadap and Magnenat-Thalmann 2001]
   → Hair medium governed by fluid-like equations
   - Macroscopic, intrinsic interaction model
   - No discontinuities
Fibers assemblies: Previous work

Main motivation
Hair simulation in Computer Graphics

Three families of models

- **Wisp-based** (or fiber-based) [Plante et al. 2001]
  → A set of strands primitives combined with a simple interaction model
Main motivation

Hair simulation in Computer Graphics

Three families of models

1. **Wisp-based** (or fiber-based) [Plante et al. 2001]
   - A set of strands primitives combined with a simple interaction model

2. Allows for fine-grain simulations [Selle et al. 2008]
Fibers assemblies: Previous work

Main motivation
Hair simulation in Computer Graphics

Three families of models

1. **Wisp-based** (or fiber-based) [Plante et al. 2001]
   - A set of strands primitives combined with a simple interaction model
   - Allows for fine-grain simulations [Selle et al. 2008]
   - Lack of stability if penalties used
   - Many contacts omitted → lack of volume
   - No dry friction (viscous model)
Fibers assemblies: Previous work

Main motivation
Hair simulation in Computer Graphics

Three families of models
③ Mixed of the two others [Mc Adams et al. 2009]
→ A mixed Eulerian-Lagrangian contact formulation
Main motivation
Hair simulation in Computer Graphics

Three families of models

- Mixed of the two others [Mc Adams et al. 2009]
  - A mixed Eulerian-Lagrangian contact formulation
  - Global volume preservation together with detailed features
Main motivation

Hair simulation in Computer Graphics

Three families of models

1. Mixed of the two others [Mc Adams et al. 2009]
   - A mixed Eulerian-Lagrangian contact formulation
   - Global volume preservation together with detailed features
   - Still no dry friction
In contrast, dry friction has been considered for a long time in Computer Graphics for the simulation of rigid bodies.
Frictional contact: Previous work

Ideal model for frictional contact
Non-penetration + Coulomb friction

\[ \mathbf{u}_N \geq 0 \]

\[ (\mathbf{u}, \mathbf{r}) \in C(\mathbf{e}, \mu) \]
Ideal model for frictional contact
Non-penetration + Coulomb friction

Most robust approach
→ Global formulation where velocities and contact forces are unknown
Implicit constrained-based methods, in practice

**Common approximation in Computer Graphics**

- Linearization of the Coulomb friction cone

→ Formulation of a Linear Complementarity Problem (LCP)
Common approximation in Computer Graphics

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→ Formulation of a Linear Complementarity Problem (LCP)

- A bunch of solvers available
Implicit constrained-based methods, in practice

**Common approximation in Computer Graphics**
- **Linearization** of the Coulomb friction cone
- → Formulation of a **Linear Complementarity Problem (LCP)**

- 😊 A bunch of solvers available
- 😞 Important drift when using too few facets
- 😞 Increasing the number of facets results in an explosion of variables
Implicit constrained-based methods, in practice

In contrast...
In Computational Mechanics

Exact Coulomb law numerically tackled for decades
Implicit constrained-based methods, in practice

In Computational Mechanics

Exact Coulomb law numerically tackled for decades

- Main application: simulation of granulars [Moreau 1994, Jean 1999]
Contributions

- Design a **generic** Newton algorithm for **exact Coulomb friction** in fiber assemblies, relying on the Alart and Curnier functional formulation

- Identify a simple criterion for **convergence**: no over-constraining
Outline

Formulating Contact in Fiber Assemblies

A Newton Algorithm for Exact Coulomb Friction

Results and Convergence Analysis

Discussion and Future Work
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Results and Convergence Analysis

Discussion and Future Work
Kirchhoff model for thin elastic rods

- Inextensible
- Elastic bending and twist
Kirchhoff model for thin elastic rods

- Inextensible
- Elastic bending and twist

In practice, three rod models used

- Implicit mass-spring system [Baraff et al. 1998]
- Corde model [Spillmann et al. 2007]
- Super-helices [Bertails et al. 2006]
Kirchhoff model for thin elastic rods

- Inextensible
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In practice, three rod models used

- Implicit mass-spring system [Baraff et al. 1998]
- CORDE model [Spillmann et al. 2007]
- Super-helices [Bertails et al. 2006]

→ We define a generic discrete rod model:

\[ M\mathbf{v} + f = 0 \quad \text{and} \quad u = H\mathbf{v} + w \]
Fiber assembly: One-step problem

- Global system (with frictional contact):
  \[
  \begin{align*}
  M \mathbf{v} + f &= H^T \mathbf{r} \\
  \mathbf{u} &= H \mathbf{v} + \mathbf{w} \\
  (\mathbf{u}, \mathbf{r}) &\text{ satisfies the Coulomb’s law}
  \end{align*}
  \]

(1)
Fiber assembly: One-step problem

- Global system (with frictional contact):
  \[
  \begin{align*}
  \mathbf{M} \mathbf{v} + \mathbf{f} &= \mathbf{H}^\top \mathbf{r} \\
  \mathbf{u} &= \mathbf{H} \mathbf{v} + \mathbf{w} \\
  (\mathbf{u}, \mathbf{r}) &= \text{satisfies the Coulomb's law}
  \end{align*}
  \]

- Compact formulation in \((\mathbf{u}, \mathbf{r})\):
  \[
  \begin{align*}
  \mathbf{u} &= \mathbf{W} \mathbf{r} + \mathbf{q} \\
  (\mathbf{u}, \mathbf{r}) &= \text{satisfies the Coulomb's law}
  \end{align*}
  \]

where \(\mathbf{W} = \mathbf{H} \mathbf{M}^{-1} \mathbf{H}^\top\) is the Delassus operator
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Results and Convergence Analysis

Discussion and Future Work
Let $\mu \geq 0$ be the friction coefficient. We define the second-order cone $K_\mu$,

$$K_\mu = \{ \| r_T \| \leq \mu r_N \} \subset \mathbb{R}^3$$
Coulomb’s law: disjunctive formulation

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Frictional contact with Coulomb’s law ($\approx 1780$)

$$(u, r) \in C(e, \mu) \iff$$
Coulomb’s law: disjunctive formulation

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Frictional contact with Coulomb’s law ($\approx 1780$)

$$(u, r) \in C(e, \mu) \iff \begin{cases} \text{either take off} & r = 0 \text{ and } u_N > 0 \\ \text{or stick} & r \in K_\mu \text{ and } u = 0 \\ \text{or slide} & r \in \partial K_\mu \setminus 0, u_N = 0 \text{ and } \exists \alpha \geq 0, u_T = -\alpha r_T \end{cases}$$
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Coulomb’s law: functional formulation

Idea
Express Coulomb’s law as $f(u, r) = 0$ with $f$ a nonsmooth function
Coulomb’s law: functional formulation

Idea
Express Coulomb’s law as \( f(u, r) = 0 \) with \( f \) a nonsmooth function

Alart and Curnier formulation (1991)

\[
\begin{align*}
f^{AC}(u, r) &= \begin{bmatrix} f^N_{AC}(u, r) \\ f^T_{AC}(u, r) \end{bmatrix} = \begin{bmatrix} P_{\mathbb{R}^+}(r_N - \rho_N u_N) \\ P_{\mathcal{B}(0, \mu r_N)}(r_T - \rho_T u_T) \end{bmatrix} - \begin{bmatrix} r_N \\ r_T \end{bmatrix}
\end{align*}
\]

where \( \rho_N, \rho_T \in \mathbb{R}^*_+ \) and \( P_K \) is the projection onto the convex \( K \).

\((u, r) \in C(e, \mu) \iff f^{AC}(u, r) = 0\)
Nonsmooth Newton on the Alart-Curnier function

Formulation of the one-step problem

\[
\begin{aligned}
\begin{cases}
u & = Wr + q \\
\mathbf{f}^{AC}(u, r) & = 0
\end{cases}
\end{aligned}
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Nonsmooth Newton on the Alart-Curnier function

Formulation of the one-step problem

\[
\begin{align*}
\begin{cases}
  \mathbf{u} &= W \mathbf{r} + \mathbf{q} \\
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\[\Leftrightarrow f^{AC}(W \mathbf{r} + \mathbf{q}, \mathbf{r}) = \Phi(\mathbf{r}) = 0\]
Nonsmooth Newton on the Alart-Curnier function

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\begin{aligned}
\begin{cases}
u & = Wr + q \\
\mathbf{f}^{AC}(u, r) & = 0
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\]

\(\Leftrightarrow \mathbf{f}^{AC}(Wr + q, r) = \Phi(r) = 0\)

Solving method: (damped) Newton algorithm

- We minimize \(\|\Phi(r)\|^2\)
- Requires the computation of \(\nabla \Phi\) (subgradients)
- Natural stopping criterion: \(\frac{1}{2} \|\Phi(r)\|^2 < \varepsilon\)
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Results
In theory...

- No proof of existence of a solution to the one-step problem
- No proof of convergence (nonsmooth function)
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- No proof of existence of a solution to the one-step problem
- No proof of convergence (nonsmooth function)

In practice

- Our fiber problems are likely to possess a solution [Cadoux 2009]
- We found an empiric criterion for convergence
• Let us define $\nu = \frac{3 \, n_{\text{contacts}}}{n_{\text{dofs}}}$
Convergence analysis

- Let us define $\nu = \frac{3n_{\text{contacts}}}{n_{\text{dofs}}}$
- Note that if $\nu > 1$ (over-constrained system), $W$ is singular
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Even quadratic convergence in favorable cases.
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- In practice, reasonable convergence properties when $\nu \leq 1$
- Even quadratic convergence in favorable cases
- Slow (or no) convergence when $\nu > 1$ (over-constrained systems)
Convergence time (in seconds) function of $\nu$

Convergence illustration

convergence time (spaghetti 31/16)

convergence time (spaghetti 76/16)
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Better suited for assemblies of compliant models than rigid bodies.
Convergence analysis

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→ $\nu$ plays the role of a conditioning number for our problem

→ better suited for assemblies of compliant models than rigid bodies

→ for over-constrained systems, a splitting strategy seems more appropriate
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Discussion and Future Work
Conclusions

Contributions

- A generic Newton solver for capturing *exact Coulomb friction* in fibers
  Relying on the Alart and Curnier functional formulation
- A simple criterion for *convergence*
  Based on the degree of *constraining* of the system
Conclusions

Contributions

• A generic Newton solver for capturing exact Coulomb friction in fibers
  Relying on the Alart and Curnier functional formulation

• A simple criterion for convergence
  Based on the degree of constraining of the system

Source code
The source code for our solver is freely available on
Limitations and Future work

Limitations

- Slow (or no) convergence for over-constrained systems
- Does not scale up well (tens to hundreds fibers vs. thousands fibers)

Future work

- Design a robust solver for thousands densely packed rods
- Carefully validate the (hair) collective behavior against real experiments
- Build a macroscopic model for fibrous media (nonsmooth laws)
Limitations and Future work

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Recent advance

Follow-up

- An improved functional formulation for exact Coulomb friction
- A splitting algorithm dedicated to large hair problems
  → In practice, this modified solver works very well for complex scenarios
Acknowledgments

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Thank You for your attention!